Equivalence regime for magnetohydrodynamic and lossy electromagnetic waves

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It is shown that for particular combinations of conductivity and frequency, magnetohydrodynamic waves have the same properties as electromagnetic waves. These conditions prevail in the ionosphere for low frequency waves, so this result may be useful in calculating loss, reflection, and transmission coefficient for magnetohydrodynamic waves. [S1063-651X(98)05101-0]

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I. INTRODUCTION

Magnetohydrodynamic (MHD) waves are important in understanding widespread dynamic phenomena. Applications include not only dynamics of the earth's ionosphere [1], but range from processes that occur within the Jovian system [2], to possible neutrino oscillations within the sun [3]. On earth, MHD waves are instrumental in understanding long period geomagnetic fluctuations, and changes in the electron density of the ionosphere.

A correlation between magnetic field fluctuations and ionospheric changes was first recorded by Rishbeth and Garriott [4], who interpreted ionospheric changes as giving rise to observed shifts in the phase of radio waves that bounced off of the ionosphere. They interpreted the changes in the ionosphere as being related to a dynamo effect, or as being driven by magnetohydrodynamic (MHD) waves. A subsequent analysis by Fraser-Smith [5] confirmed a relationship between geomagnetic fluctuations, sunspot activity, and ionospheric changes for oscillations with periods greater than two days. During the next few years a model emerged which pictures an Alfvèn, or MHD, wave as a standing wave, the boundaries of which are the ionosphere in the northern and southern hemispheres. In other words, the MHD wave travels along the ambient magnetic field line of the earth, and the ionosphere acts as a very good reflector in each hemisphere. Southwood [6] and Chen and Hasegawa [7] showed that the solar wind can excite shear Alfvèn waves at the magnetopause, which leads to the standing MHD waves described above.

A detailed examination of the relation between electromagnetic waves in the ionosphere and MHD waves was undertaken by Hughes [8]. Following Dungey's work [9], Hughes broke the problem into two parts: One part is a formulation which contains a vertical current, and one part has no vertical current. He showed that the part without a vertical current produces a magnetic field component at ground level, while the other part is negligible in comparison. This work was extended by Hughes and Southwood [10]. Using radio waves and a geostationary satellite, Davies and Hartmann [11] observed periodic fluctuations in the electron concentration in the ionosphere. These observations were verified and extended by Okuzawa and Davies, who also considered the spectral characteristics of the fluctuations [12]. During the 1980s there were a large number of publications that cemented the view that MHD waves and geomagnetic fluctuations were intimately related [13–16].

Another important result was that of Pool and Sutcliffe in 1987 [17]. In this work they derived a quantitative relation between the change in the electron concentration of the ionosphere and magnetic field fluctuations observed on the ground. Although the work contained many approximations, it demonstrated the overall validity of the idea that changes in the electron concentration can be related to geomagnetic fluctuations.

The discussion presented above gives only an indication of the widespread application of MHD waves. A problem with MHD models, however, is the complexity of the coupled partial differential equations, and numerical methods are often used. In this paper it is shown that for values of the frequency and conductivity used in many of the references cited above, there are situations in which an analysis of an electromagnetic wave can yield information about properties of physical interest. We do not claim that the full MHD analysis can always be avoided, rather we show that there are some situations where an analysis of the electromagnetic wave by itself is sufficient to uncover physical properties of interest.

II. WAVE SOLUTIONS

A. Magnetohydrodynamic waves

Magnetohydrodynamic waves may be excited whenever there is a conducting fluid permeated with an external magnetic field. To see how they arise, one may consider a neutral parcel of the fluid of mass density ρ, velocity $\boldsymbol{v}$, and current density $\boldsymbol{j}$. The force law $\boldsymbol{F}=m\boldsymbol{a}$ for this parcel becomes

$$\rho\frac{d\boldsymbol{v}}{dt}=\frac{\boldsymbol{j}\times\boldsymbol{B}}{c}-\boldsymbol{\nabla}p. \tag{1}$$

MHD waves are governed by Eq. (1), the conservation law

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\rho\boldsymbol{v}=0, \tag{2}$$

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and the Maxwell equations

$$\nabla\times\boldsymbol{E}=-\dot{\boldsymbol{B}}/c \tag{3}$$

and

$$\nabla\times\boldsymbol{B}=4\pi/c\boldsymbol{j}, \tag{4}$$

where the overdot stands for the time derivative, and the term $\dot{\boldsymbol{E}}/c$ is neglected due to the low frequency of MHD waves. The final ingredient is Ohm's law, which, due to the fact that the fluid is moving, is $\boldsymbol{E}=(\boldsymbol{j}/\sigma)-\boldsymbol{v}\times\boldsymbol{B}/c$. For good conductors, it is sometimes possible to use this equation in the limit of infinite conductivity, but we are interested in the more general case of noninfinite conductivity. It is further assumed that the conductivity is constant and a scalar quantity.

The variables are expanded in terms of small changes according to

$$\rho=\rho_0+\rho_1, \tag{5}$$

$$\boldsymbol{B}=\boldsymbol{B}_0+\boldsymbol{b}. \tag{6}$$

In the above, $\boldsymbol{B}_0$ is the uniform background field of the earth, and $\boldsymbol{b}$ represents a small fluctuation due to the MHD wave. Similarly, ρ_0 represents the density in the absence of the wave, and ρ_1 is the perturbation due to the MHD wave. The velocity of the fluid, $\boldsymbol{v}$, results from the MHD wave, and is therefore assumed to be of order b and ρ_1. Using these last two expansions in the equations above, one obtains

$$\frac{\partial\rho_1}{\partial t}+\nabla\cdot\rho_0\boldsymbol{v}=0, \tag{7}$$

$$\rho_0\frac{d\boldsymbol{v}}{dt}=-s^2\nabla\rho_1+\frac{1}{4\pi}(\nabla\times\boldsymbol{b})\times\boldsymbol{B}_0, \tag{8}$$

and

$$\dot{\boldsymbol{b}}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B}_0)-\frac{c}{\sigma}\nabla\times\boldsymbol{j}, \tag{9}$$

where s is the speed of sound. The speed of sound arises from the pressure gradient in Eq. (1), and is given by $s^2=\gamma p_0/\rho_0$ where p_0 and ρ_0 are the ambient pressure and density, and γ is the ratio of specific heats. Details may be found elsewhere [18].

Taking the curl of Eq. (4), and using that in Eq. (9), one obtains

$$\dot{\boldsymbol{b}}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B}_0)+\frac{c^2}{4\pi\sigma}\nabla^2\boldsymbol{b}. \tag{10}$$

Finally, taking the time derivative of Eq. (8), and using Eq. (10) in that result, one obtains

$$\rho_0\ddot{\boldsymbol{v}}=\frac{1}{4\pi}\left[\nabla\times\left(\nabla\times[\boldsymbol{v}\times\boldsymbol{B}_0]+\frac{c^2}{4\pi\sigma}\nabla^2\boldsymbol{b}\right)\right]\times\boldsymbol{B}_0 +s^2\rho_0\nabla(\nabla\cdot\boldsymbol{v}). \tag{11}$$

There are several plane wave type solutions depending on the orientation of the wave vector $\boldsymbol{k}$ and the velocity of the fluid $\boldsymbol{v}$. Here we shall consider the special case of transverse waves, for which $\boldsymbol{k}\cdot\boldsymbol{v}=0$. For this case the solutions to the above equations may be shown to be

$$\boldsymbol{v}=v_x\hat{\boldsymbol{x}}e^{i(kz-\omega t)}, \tag{12}$$

$$\boldsymbol{b}=-\frac{v_xB_0k/\omega}{\left(1+i\dfrac{c^2k^2}{4\pi\omega\sigma}\right)}\hat{\boldsymbol{x}}e^{i(kz-\omega t)}, \tag{13}$$

where $\boldsymbol{k}=k\hat{\boldsymbol{z}}$, the real part of the right hand side is implied in the two equations above, and

$$k^2v_A^2=\omega^2\left(1+i\frac{c^2k^2}{4\pi\omega\sigma}\right) \tag{14}$$

where $\boldsymbol{v}_A\equiv\boldsymbol{B}_0/\sqrt{4\pi\rho}$ is the Alfvèn velocity. From Eq. (14) it is seen that in the limit that $\sigma\to\infty$, v_A is the phase velocity of the wave. However, for finite conductivity, v_A is not the phase velocity, and, moreover, the dispersion relation (14) gives rise to a frequency dependent group velocity.

The dispersion relation also shows that the wave vector is complex, so we write it as $k=\beta+i\alpha$. Using this, and $\tau=c^2/4\pi\sigma v_A^2$, we find

$$\beta=\frac{\omega}{\sqrt{2}v_A}\left(\frac{\sqrt{1+(\omega\tau)^2}+1}{1+(\omega\tau)^2}\right)^{1/2} \tag{15}$$

$$\alpha=\omega^2\tau/\sqrt{2}v_A\{[1+(\omega\tau)^2][1+\sqrt{1+(\omega\tau)^2}]\}^{-1/2}.$$

As expected, the noninfinite conductivity causes attenuation and the plane waves propagate as $e^{-\alpha z}e^{i(\beta x-\omega t)}$. The group velocity $v_g=d\omega/d\beta$ turns out, after some manipulation, to be

$$v_g=\omega/\beta\left(1-\frac{(\tau\omega)^2}{1+(\tau\omega)^2}+\frac{\tau^2\omega^6/4\beta^2v_A^2[1+(\tau\omega)^2]}{2\beta^2v_A^2[1+(\tau\omega)^2]-\omega^2}\right)^{-1}. \tag{16}$$

B. Electromagnetic waves in a conducting medium

Now we consider conventional plane electromagnetic waves propagating in a medium with conductivity σ. The magnitudes of the fields are given by

$$\mathcal{E}=Ee^{i(kx-\omega t)}, \tag{17}$$

$$\mathcal{B}=Be^{i(kx-\omega t)}, \tag{18}$$

where E and B are the amplitudes for the electric and magnetic fields. For plane waves they are related by $B=(ck/\omega)E$, where again $k=\beta+i\alpha$, but now

$$\beta=\frac{\omega}{c}\left(\frac{\sqrt{1+(4\pi\sigma/\omega)^2}+1}{2}\right)^{1/2}, \tag{19}$$

$$\alpha=\frac{\omega}{c}\left(\frac{\sqrt{1+(4\pi\sigma/\omega)^2}-1}{2}\right)^{1/2}.$$

III. EQUIVALENCE

These results underscore the difference between MHD waves and ordinary electromagnetic waves. For example, in the limit $\sigma\to\infty$, MHD waves propagate with phase velocity

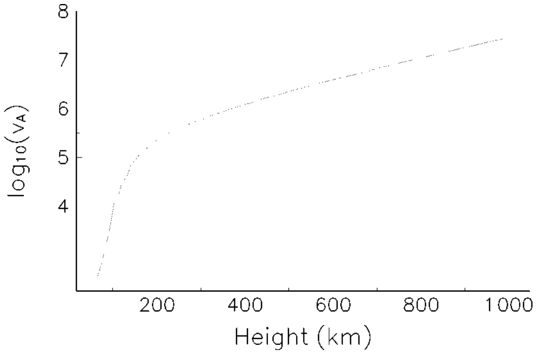

FIG. 1. The $\log_{10}$ of Alfvèn velocity in cm s^{-1} as a function of altitude. The density is taken from Ref. [20] and B_0 was taken as 0.5 G.

ω/k, while the attenuation constant α goes to zero. For electromagnetic waves, however, in this limit the attenuation constant becomes infinite and waves cannot propagate. In the other limit that $\sigma \to 0$, electrodynamic waves propagate with phase velocity ω/k and the attenuation constant α goes to zero, while for MHD waves the wave vector goes to zero.

Nevertheless, there is an important regime in which these waves have the same dispersion relation. Consider low frequency waves propagating in the ionosphere. Typical frequencies are of the order of $\omega \sim 1$, and may vary a few orders of magnitude either way. In the ionosphere, the conductivity is really a tensor quantity due to the preferred direction imposed by the ambient magnetic field. For MHD waves that travel predominantly along these field lines, the important component of the conductivity is the transverse, or Pederson, conductivity, which has a maximum value between 10^5 and 10^6 s^{-1} (see Ref. [8]).

In any event, consider the case that

$$1 \ll 4\pi\sigma/\omega \ll c^2/v_A^2. \tag{20}$$

To help understand the limits of validity, the Alfvèn velocity is displayed as a function of altitude in Fig. 1.

This shows that there is a significant range of ω over which Eq. (20) is satisfied. For example, the conductivity is at its maximum value around 300 km, and Eq. (20) becomes, approximately, with $\omega = 2\pi\nu$, and using $v_A = 6 \times 10^5$ cm/s at 300 km,

$$10^6 \gg \nu \gg 10^{-3}. \tag{21}$$

This result shows that Eq. (20) is satisfied for a very important range of frequencies that propagate in the ionosphere. Of course there are other physical situations in which Eq. (20) is satisfied as well. The point is, using Eq. (20), the wave vector becomes

$$k = (1+i)\sqrt{2\pi\omega\sigma}/c. \tag{22}$$

This result is true for both the MHD result (15) and the electromagnetic result (19). Thus, despite the differences discussed above, there exists a range of frequency and conductivity for which electromagnetic waves and MHD waves have the same dispersion relation, velocity, and attenuation.

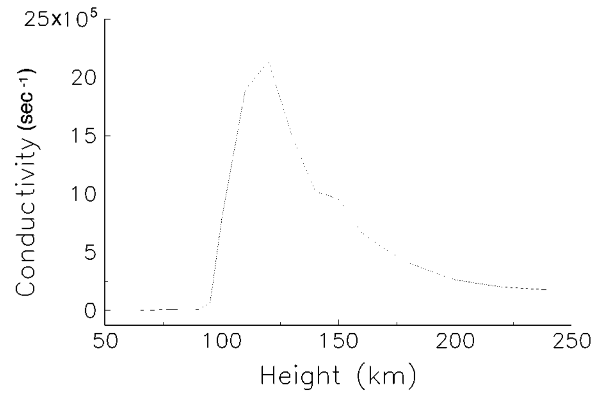

FIG. 2. The conductivity in s^{-1} vs altitude in km. The peak conductivity is about 25×10^5 s^{-1}.

IV. AN APPLICATION

One of the problems of general interest with regard to earthly MHD waves concerns the amount of energy that penetrates the ionosphere; how much is reflected back into the magnetosphere, and how much is absorbed. This problem was investigated in detail, for example, in Ref. [8].

The equivalence established above can be used to solve this problem in a simple fashion. The idea is to compute the reflectivity of the ionosphere as an MHD wave penetrates it from above. However, in the solutions presented above it was assumed that the conductivity and Alfvèn velocity were constant. Figure 1 shows that this is not true in the ionosphere, so to study wave propagation there one must resort to other methods. One such technique is to break the region into slabs of uniform conductivity, apply the boundary conditions at each interface, and from that model calculate all quantities of interest, as we did recently for electromagnetic waves [19]. For electromagnetic waves, since there are no current or charge densities at the interfaces, the boundary conditions are the continuity of the fields. In general, an acoustic wave will impose the additional conditions that, for small amplitude oscillation, the normal component of the fluid velocity and the pressure are continuous. For transverse waves the velocity condition is automatically satisfied; moreover, transverse MHD waves do not have pressure variations [as Eq. (7) implies] if ρ_0 is constant. Therefore, if one is considering propagation of a transverse MHD wave through a medium with varying conductivity, one may analyze the problem approximately by partitioning the medium into slabs of uniform conductivity, and use the results concerning attenuation, reflectivity and transmission obtained for electromagnetic waves (of Ref. [19]). Due to the equivalence of wave vectors, and the fact that all boundary conditions are satisfied as discussed above, one may say that, as far as attenuation, reflectivity, and transmission coefficients are concerned, a transverse MHD wave is equivalent to an electromagnetic wave.

Now, in Ref. [19], we modeled the conductivity of the ionosphere with a Gaussian form. Here we can do better, and use an actual conductivity profile of the ionosphere, as shown in Fig. 2.

A note about the conductivity displayed in Fig. 2 is in order. This represents the Pederson conductivity, which is the conductivity in a direction parallel to the electric field and perpendicular to the magnetic field. It is obtained using actual data from the literature [20] combined with theoretical values for the collision frequency of the charged particles [21]. Reference [21] contains the details on how the actual expression was derived. Finally, it should be noted that the actual value of the electron density, which is one of the empirical parameters given in Ref. [20], depends on whether it is day or night, and on sunspot activity. The result displayed in Fig. 2 is for daytime, with sunspot activity.

For the current application, we will consider only the reflection coefficient, R, which represents the ratio of the power incident on the ionosphere to that reflected. To model the ionosphere, consider breaking it into N slabs. In a given region the fields are given by, suppressing the time dependence and vector nature,

$$\mathcal{E}_n = E_n e^{ik_n x} + E_n' e^{-ik_n x}, \tag{23}$$

$$\mathcal{B}_n = B_n e^{ik_n x} - B_n' e^{-ik_n x}, \tag{24}$$

where E_n' and B_n' are the magnitudes of the reflected waves. It can be shown that the relation between the electric field in region $n+1$ and the field in region n is given by

$$E_{n+1} = \frac{e^{ik_{n+1}x_n}}{2}\left[\left(1+\frac{k_n}{k_{n+1}}\right)E_n e^{-ik_n x_n} + \left(1-\frac{k_n}{k_{n+1}}\right)E_n' e^{ik_n x_n}\right]. \tag{25}$$

This result is a direct generalization of the result established in Ref. [19], except here the interface is located at x_n, and the slabs are not equally spaced.

The reflection coefficient is defined as

$$R = E_{N+1}' E_{N+1}'^{*} / E_{N+1} E_{N+1}^{*}, \tag{26}$$

and may be computed using Eq. (25). The details can be found in Ref. [19], and here a result for R will be given. Using this formulation, and the equivalence established above, one finds the reflection coefficient to be $R=99.8\%$, which is in general agreement with the results of Ref. [8], which quotes $R \approx 99\%$.

V. DISCUSSION AND NEXT STEPS

In general, MHD waves and electromagnetic waves are quite different. They have different attenuation, different wave vectors, different phase and group velocities. In the limit of large or small conductivity, MHD and electromagnetic waves behave in the opposite manner (as explained in Sec. III). In general, MHD waves will impose additional boundary conditions across an interface. Physically, MHD waves incorporate fluid motion, along with Newton's law, and propagate in the presence of an external field, while electromagnetic waves do not. *Yet, despite all of these differences, it has been shown that for situations of interest, there is a kind of equivalence between these waves.* In particular it was shown that the wave numbers and boundary conditions are the same when Eq. (20) is satisfied for transverse waves.

As an application of this result, we calculated the reflection coefficient of an MHD wave incident on the ionosphere from above. We obtained a value $R=99.8\%$, in agreement with known results.

A limitation to these results concerns nontransverse waves propagating through a nonuniform medium, which are of physical interest for the ionosphere. If the propagation is analyzed by breaking the region into thin uniform slabs, then the MHD wave imposes additional boundary conditions. In this case the normal component of the velocity and the pressure (which is not constant) must be continuous at each boundary. Finally, we should reformulate equations in terms of the tensor conductivity, since that is what prevails in the ionosphere. This is especially important with respect to the directions of the fields; however, for calculations of attenuation and reflectivity, we expect the results here to be essentially correct. Future work will investigate these areas.

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